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$$\begin{aligned}
\therefore a(R_1 + R_2) &= \frac{av^2}{g} [\sin 2\alpha + \sin 2(\alpha - \beta)] = \frac{2av^2}{g} \sin 2(\alpha - \beta) \cos \beta \\
&= \frac{2av^2}{g} [\sin \alpha \cos(\alpha - \beta) + \cos \alpha \sin(\alpha - \beta)] \cos \beta \\
&= \frac{a^4}{c^2} \left( \frac{\sin \alpha \cos(\alpha - \beta) + \cos \alpha \sin(\alpha - \beta)}{\cos \alpha \sin(\alpha - \beta)} \right). \\
R_1 R_2 &= \frac{v^4}{g^2} \sin 2\alpha \sin 2(\alpha - \beta) = \frac{4v^4}{g^2} \sin \alpha \cos \alpha \sin(\alpha - \beta) \cos(\alpha - \beta) \\
&= \frac{a^4 \sin \alpha \cos(\alpha - \beta)}{c^2 \cos \alpha \sin(\alpha - \beta)}. \\
\therefore a(R_1 + R_2) - R_1 R_2 &= \frac{a^4 \cos \alpha \sin(\alpha - \beta)}{c^2 \cos \alpha \sin(\alpha - \beta)} = \frac{a^4}{c^2}.
\end{aligned}$$

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## NUMBER THEORY AND DIOPHANTINE ANALYSIS.

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Edited by Dr. G. E. Wahlin, University of Illinois.

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183. Proposed by MERTON T. GOODRICH, Dixfield, Maine.

What relations must exist between the quantities  $A$ ,  $B$ , and  $C$  in the harmonic ratio  $\frac{AB}{(A+B+C)(-C)} = -1$  so that they will be positive integers.

II. Solution by the PROPOSER.

Solving the given equation for  $A$ , we have  $A = \frac{(B+C)C}{B-C}$ . Since  $A$ ,  $B$ , and  $C$  are positive, we see that  $B-C$  must be positive. Since  $A$  is an integer, either  $\frac{C}{B-C} = \frac{M}{N} = K$ , or  $\frac{B+C}{B-C} = \frac{M'}{N'} = K'$ , where  $M$  and  $N$ ,  $M'$  and  $N'$  are integers, and  $M$  prime to  $N$ , and  $M'$  prime to  $N'$ . From the first of these equations,  $C = K(B-C)$ . Since  $C$  and  $B-C$  are positive,  $K$  must be positive. Solving this last equation for  $B$ , we have  $B = \frac{(K+1)C}{K} = \frac{(M+N)C}{M}$ . Since  $M$  and  $N$  are relatively prime,  $M+N$  is prime to  $M$ . Hence,  $B$  being an integer,  $C$  is divisible by  $M$ . That is,  $C = MD' = MND'/N = KD$ , where  $D = ND'$ , and  $D'$  and hence  $D$  are positive integers.

Substituting  $KD$  for  $C$  in the expression for  $B$ ,  $B = (K+1)D = C + D$ . Substituting  $KD$  for  $C$  and  $(K+1)D$  for  $B$  in the expression for  $A$ ,  $A = (2K+1)KD = (2K+1)C$ . Putting  $\frac{M}{N}$  for  $K$ ,  $A = \frac{(2M+N)MD}{N^2}$ . This tells us that if  $N$  is odd  $D = N^2 \cdot \lambda$ ; but if  $N$  is even,  $2M+N$  is even and then  $D = \frac{N^2 \cdot \lambda}{2}$ . Hence we have this set of relations:  $A = (2K+1)C$ ,  $B = C + D$ ,

$C=KD$ , where  $K$  is a positive fraction in its lowest terms or a positive integer, and  $D$  is a positive integer as above determined.

The second condition is not independent of the first, because, if  $K'$  is substituted for  $2K+1$  and  $2D'$  for  $D$ , in the above set of relations, then we have

$$\frac{B+C}{B-C} = \frac{D'K' + D + D'K' - D'}{D'K' + D' - D'K' + D'} = \frac{2D'K'}{2D'} = K',$$

which is the second condition. Hence the set of values found above includes all the possible relations which make  $A$ ,  $B$ , and  $C$  positive integers. The values of  $A$  and  $B$  may be interchanged, and  $A$ ,  $B$ , and  $C$  may each be multiplied by a common factor without changing the value of the original ratio.

Also solved by A. H. Holmes.

184. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that  $\frac{\pi}{12} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{30} + \tan^{-1} \frac{1}{112} + \dots$ , where 2, 8, 30, 112..., is a recurring series with the recursion formula  $u_n = 4u_{n-1} - u_{n-2}$ .

Solution by the PROPOSER.

If we reduce  $\sqrt{3}$  to a continued fraction, we get for the convergents,

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \dots$$

The alternate convergents,

$$\frac{1}{1}, \frac{5}{3}, \frac{19}{11}, \frac{71}{41}, \frac{265}{153}, \dots$$

are formed by taking the ratios of the corresponding terms of the two recurring series

$$\begin{array}{l} 1, 5, 19, 71, 265, \dots \\ 1, 3, 11, 41, 153, \dots \end{array}$$

both having the same scale of relation  $u_n = 4u_{n-1} - u_{n-2}$ .

We find by the usual methods that the  $n$ th terms of the two series are

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = u_n + u_{n-1}, \text{ and } \frac{\alpha^n - \beta^n}{\alpha - \beta} - \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = u_n - u_{n-1},$$